EXISTENCE OF DIVERGENT BIRKHOFF NORMAL FORMS OF HAMILTONIAN FUNCTIONS

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1. Introduction

Let $\mathbf{R}^{2n}$ be the standard symplectic space, equipped with the symplectic two-form $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. Let $h(x,y) = O(2)$ be a real analytic functions defined near $0 \in \mathbf{R}^{2n}$. Under suitable non-degeneracy condition on the quadratic form of h, one may find linear symplectic coordinates so that

$$h(x,y) = \frac{1}{2}\sum_{1\leq j\leq\kappa} i\lambda_j(x_j^2+y_j^2) + \frac{1}{2}\Re\sum_{\kappa<k,l\leq n,\lambda_l=\overline{\lambda}_k} \lambda_k(x_k+ix_l)(y_k-iy_l)+O(3), \tag{1.1}$$

where $\kappa = 0,\ldots,n$, and λ_j is pure imaginary precisely when $1 \leq j \leq \kappa$, and $\lambda_1, -\lambda_1, \ldots, \lambda_n, -\lambda_n$ are eigenvalues of $H_{zz}(0)J$ with $z = (x,y)$ and $Jx_j = y_j = -J^2y_j$. One says that $\lambda_1,\ldots,\lambda_n$ are *non-resonant*, if $\lambda\cdot\alpha \equiv \lambda_1\alpha_1+\cdots+\lambda_n\alpha_n \neq 0$ for all multi-indices of integers $\alpha \neq 0$. The Birkhoff normal form says that under the non-resonance condition on λ, there is a formal symplectic transformation of $\mathbf{R}^{2n}$ sending h into $\hat{h}$ that is a real formal power series in $x_j^2+y_j^2$ $(1 \leq j \leq \kappa)$, $(x_k+ix_l)(y_k-iy_l)$ $(\kappa < k, l \leq n)$. Notice that, up to the order of $\lambda_1,\ldots,\lambda_n,-\lambda_1,\ldots,-\lambda_n$, the Birkhoff normal form $\hat{h}$ is independent of the choice of the normalizing transformations. In [12], Siegel showed that the Birkhoff normal form cannot be realized by convergent symplectic transformations in general. In fact, Siegel [13] showed that when $\kappa = n \geq 2$, for a real analytic function with any prescribed nonresonant $\lambda_1,\ldots,\lambda_n$ and with generic higher order terms, there exists no convergent normalizing transformation.

Despite Siegel's divergence results and many other results, a basic question, which remains unsettled until now, is if there exists a divergent Birkhoff normal form arising from a real analytic function. This question was pointed out by Eliasson [2]. To the author's knowledge, there seems no example of divergent normal form in other normal form problems in the literature. The divergence of Birkhoff normal form implies, of course, that of all normalizing transformations of the given function. The importance of such a divergent normal form was demonstrated by Pérez-Marco [9] very recently.

In this paper we shall prove

Theorem 1.1. *Let $\kappa = 0,\ldots,n$ and $n \geq 2$. Assume that $\kappa \neq 1$ when $n = 2$. There exists a divergent Birkhoff normal form of some analytic real function (1.1), defined near $0 \in \mathbf{R}^{2n}$ and having non-resonant $\lambda_1,\ldots,\lambda_n$.*

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It is necessary to exclude the case of non-real λ_2/λ_1 in the theorem when $n = 2$. Indeed, by a theorem of Moser [8], the Birkhoff normal form is always realized by some convergent transformation when $n = 2$ and λ_2/λ_1 is not real. One can see, from the proof of the theorem, that the set of real analytic Hamiltonian functions with divergent Birkhoff normal form is dense in a suitable topology. One may also apply a result of Pérez-Marco [9] and the above theorem to conclude that generic Hamiltonian functions with the above quadratic form have divergent normal form too.

For the Birkhoff normal form theory, the reader is referred to, besides the above mentioned references, papers of Moser [7], Rüssmann [10], [11], Brjuno [1], Vey [15], Ito [6], Stolovitch [14], Giorgilli [3], and the author [4], [5]. Papers by Brjuno [1] and by Pérez-Marco [9] contain extensive references also.

The proof of Theorem 1.1 is based on the method of small divisors. One would expect that the present approach will have implications for other small-divisor problems. We will however focus on the Hamiltonian functions, to demonstrate how the small-divisors enter the normal form.

2. Proof of the theorem

We may restrict ourselves to $n = 2$, since the sought h for higher dimension can be obtained trivially by adding suitable quadratic terms.

Consider a real analytic (real-valued) function

$$h(x, y) = \sum_{j=1}^{2} \lambda_j x_j y_j + O(3),$$

where λ_1, λ_2 are non-resonant. Let $S(x, \hat{y})$ be a real analytic function defined near $0 \in \mathbf{R}^2 \times \mathbf{R}^2$ with $S(x, \hat{y}) = O(d)$, $d > 2$. Let $\varphi\colon (x, y) \to (\hat{x}, \hat{y})$ be a symplectic map defined by

$$\hat{x}_j = x_j - S_{\hat{y}_j}(x, \hat{y}), \quad \hat{y}_j = y_j + S_{x_j}(x, \hat{y}), \quad j = 1, 2. \tag{2.1}$$

Note that

$$\varphi\colon \hat{x}_j = x_j - S_{y_j}(x, y) + O(d), \quad \hat{y}_j = y_j + S_{x_j}(x, y) + O(d).$$

Put $\hat{h} = h \circ \varphi^{-1}$. Then $h(x, y) = \hat{h}(\hat{x}, \hat{y})$ has the expansion

$$\hat{h}(x, y) + \sum \lambda_j (x_j S_{x_j}(x, y) - y_j S_{y_j}(x, y)) + O(d + 1).$$

Define the projection

$$\mathcal{N} \sum_{\alpha\beta} h_{\alpha\beta} x^\alpha y^\beta = \sum_{\alpha} h_{\alpha\alpha} x^\alpha y^\alpha.$$

Note that h is in a Birkhoff normal form, if and only if $\mathcal{N}h$ agrees with h. For the special case of $h = \mathcal{N}h + O(d)$ with $d \geq 3$, taking

$$S_{\alpha\beta} = \frac{1}{\lambda \cdot (\alpha - \beta)} h_{\alpha\beta}, \quad |\alpha| + |\beta| = d, \quad \alpha \neq \beta$$

yields $\hat{h} = \mathcal{N}\hat{h} + O(d+1)$. In the above and in what follows α, β stand for multi-indices of non-negative integers. We also write $|\alpha - \beta| = |\alpha_1 - \beta_1| + |\alpha_2 - \beta_2|$. In general, inductively

one finds

$$S_{\alpha\beta} = \frac{1}{\lambda\cdot(\alpha-\beta)}\{h_{\alpha\beta}+Q_{\alpha\beta}(h)\}, \quad \alpha\neq\beta, \tag{2.2}$$

so that $\hat{h}_{\alpha\beta}=0$ for $\alpha\neq\beta$, i.e., so that φ, a formal symplectic map of $\mathbf{R}^4$, transforms h into the Birkhoff normal form $\hat{h}$. Notice that the above expression $Q_{\alpha\beta}(h)$ stands for a polynomial (with integer coefficients) in quantities

$$h_{\alpha'\beta'}, \frac{1}{\lambda\cdot(\alpha''-\beta'')}; \alpha''\neq\beta'', \max\{|\alpha|'+|\beta'|, |\alpha''|+|\beta''|\} < |\alpha|+|\beta|.$$

Note that $\lambda_1 = h_{1,0,1,0}, \lambda_2 = h_{0,1,0,1}$. One also has

$$\hat{h}_{\alpha\alpha} = h_{\alpha\alpha} + D_{\alpha\alpha}(h), \tag{2.3}$$

where $D_{\alpha\alpha}(h)$ is a polynomial in quantities

$$h_{\alpha'\beta'}, \frac{1}{\lambda\cdot(\alpha''-\beta'')}; \quad \alpha''\neq\beta'', \max\{|\alpha|'+|\beta'|, |\alpha''|+|\beta''|\} < |\alpha|+|\beta|.$$

We need to know more about the term $D_{\alpha\alpha}$ in (2.3).

Lemma 2.1. *Let $S(x,\hat{y})$ be a power series starting with terms of order d, and let $T=[S]_d$ be the sum of all monomials in S of order $d>2$. Let φ be the mapping defined by (2.1). Let $\hat{h}=h\circ\varphi^{-1}$. Assume that $h_{\alpha\beta}=0$ for $|\alpha|+|\beta|<d$ and $\alpha\neq\beta$, and that $\hat{h}_{\alpha\beta}=0$ for $|\alpha|+|\beta|<2d-1$ and $\alpha\neq\beta$. Then*

$$\hat{h}(x,y)-\mathcal{N}h(x,y) = \mathcal{N}\{\sum_{j,k=1}^{2}\lambda_j(\frac{1}{2}T_{x_j}T_{y_j}+y_jT_{y_jy_k}T_{x_k}-x_jT_{x_jy_k}T_{x_k})\}+O(2d-1). \tag{2.4}$$

Proof. Returning to (2.1), we get

$$\hat{x}_j = x_j - S_{y_j}(x,y) - \sum_{k=1}^{2} T_{y_jy_k}(x,y)T_{x_k}(x,y) + O(2d-2),$$
$$\hat{y}_j = y_j + S_{x_j}(x,y) + \sum_{k=1}^{2} T_{x_jy_k}(x,y)T_{x_k}(x,y) + O(2d-2).$$

Now

$$\begin{aligned} h(x,y) = \hat{h}(\hat{x},\hat{y}) &= \sum_{j,k=1}^{2}\lambda_j(x_jT_{x_jy_k}T_{x_k} - y_jT_{y_jy_k}T_{x_k} - \frac{1}{2}T_{x_j}T_{y_j}) \\ &+ \hat{h}(x,y) + \sum \alpha_j\hat{h}_{\alpha\alpha}x^{\alpha-\delta_j}y^{\alpha-\delta_j}(x_jS_{x_j}-y_jS_{y_j}) + O(2d-1), \end{aligned}$$

where $\delta_j=(0,\dots,1,\dots,0)$ with the 1 at the j-th place. Applying the projection $\mathcal{N}$ to both sides yields (2.4). □

The term $\lambda\cdot(\alpha-\beta)$ in (2.2) is the small-divisor used by Siegel in his first proof [12] for the divergence of Birkhoff's normalization for Hamiltonian functions. Notably, this small divisor, when $|\alpha-\beta|=|\alpha|+|\beta|$, does not appear in (2.3). We now identify the small-divisor that contributes to the divergence of a Birkhoff normal form.

Lemma 2.2. *Keep nations and assumptions in Lemma 2.1. Let $N+m=d$, $\alpha=(N,m-1)$, $a=(N,0)$ and $b=(0,m)$. Assume that $m\geq 1$. Then*

$$\hat{h}_{\alpha\alpha}=h_{\alpha\alpha}-\frac{m^2(\lambda_1 N-\lambda_2)(h_{ab}+Q_{ab}(h))(h_{ba}+Q_{ba}(h))}{(\lambda\cdot(a-b))^2}+\frac{h_{ab}A_{N+m}(h)+h_{ba}B_{N+m}(h)}{\lambda\cdot(a-b)}+C_{N+m}(h), \tag{2.5}$$

where A_{N+m},B_{N+m} and C_{N+m} are polynomials in $h_{\alpha'\beta'},\frac{1}{\lambda\cdot(\alpha''-\beta'')}$ with

$$\alpha''\neq\beta'',(\alpha'',\beta'')\neq(a,b),(b,a);|\alpha'|+|\beta'|\leq N+m,|\alpha''|+|\beta''|\leq N+m.$$

Proof. Write

$$T(x,y)=T_{ab}x_1^N y_2^m+T_{ba}x_2^m y_1^N+\sum_{(a',b')\neq(a,b),(b,a)}T_{a'b'}x^{a'}y^{b'}.$$

Then we obtain

$$\sum_{j,k}\lambda_j x_j T_{x_j y_k}T_{x_k}=T_{ab}T_{ba}(\lambda_1 Nm^2(x_1y_1)^N(x_2y_2)^{m-1}+\lambda_2 mN^2(x_1y_1)^{N-1}(x_2y_2)^m)+\dots,$$

$$\sum_{j,k}\lambda_j y_j T_{y_j y_k}T_{x_k}=0+\dots,$$

$$\sum_{j}\lambda_j T_{x_j}T_{y_j}=T_{ab}T_{ba}(\lambda_1 N^2(x_1y_1)^{N-1}(x_2y_2)^m+\lambda_2 m^2(x_1y_1)^N(x_2y_2)^{m-1})+\dots,$$

where the omitted terms have coefficients that are linear combinations with integer coefficients in $T_{ab}T_{a'b'}$, $T_{ba}T_{a''b''}$, and $T_{a'b'}T_{a''b''}$ with (a',b'), $(a'',b'')\neq(a,b)$, (b,a). Thus

$$\hat{h}(x,y)-h(x,y)=T_{ab}T_{ba}\{(\lambda_1-\lambda_2 m)N^2(x_1y_1)^{N-1}(x_2y_2)^m+(\lambda_2-\lambda_1 N)m^2(x_1y_1)^N(x_2y_2)^{m-1}\}+\dots.$$

Combining (2.2), we obtain (2.5). □

Proposition 2.3. *Let $h(x,y)=\lambda_1x_1y_1+\lambda_2x_2y_2+O(3)$ be a real analytic function with $0<\lambda_1<\lambda_2$. Assume that λ_1,λ_2 are non-resonant. Let φ be any formal symplectic map so that $\hat{h}(x,y)=h\circ\varphi^{-1}(x,y)$ is in the Birkhoff normal form with quadratic form $\lambda_1x_1y_1+\lambda_2x_2y_2$. Then for $\alpha=(N,m-1)$, $a=(N,0),b=(0,m)$ with $m\geq 1$, one has*

$$\hat{h}_{\alpha\alpha}=h_{\alpha\alpha}-\frac{m^2(\lambda_1 N-\lambda_2)(h_{ab}+Q_{ab}(h))(h_{ba}+Q_{ba}(h))}{(\lambda\cdot(a-b))^2}+\frac{h_{ab}A_{ab}(h)+h_{ba}B_{ab}(h)+C_{ab}(h)}{\lambda\cdot(a-b)}+\hat{Q}_{ab}(h), \tag{2.6}$$

where Q_{ab} is a polynomial in $h_{\alpha'\beta'},\frac{1}{\lambda\cdot(\alpha''-\beta'')}$ with

$$\alpha''\neq\beta'',\max\{|\alpha'|+|\beta'|,|\alpha''|+|\beta''|\}<|a|+|b|,$$

$\hat{Q}_{ab}$ is a polynomial in $h_{\alpha'\beta'},\frac{1}{\lambda\cdot(\alpha''-\beta'')}$ with

$$\alpha''\neq\beta'',(\alpha'',\beta'')\neq(a,b),(b,a),|\alpha'|+|\beta'|<2|\alpha|,|\alpha''|+|\beta''|\leq|a|+|b|$$

and A_{ab}*,* B_{ab}, C_{ab} *are polynomials in* $h_{\alpha'\beta'}$*,* $\frac{1}{\lambda\cdot(\alpha''-\beta'')}$ *with*

$$\alpha''\neq\beta'', (\alpha'',\beta'')\neq(a,b),(b,a), \max\{|\alpha'|+|\beta'|,|\alpha''|+|\beta''|\}\leq|a|+|b|.$$

Proof. We apply a symplectic map φ_1 of the form (2.1), in which

$$S(x,\hat{y})=\sum_{\alpha\neq\beta,3\leq|\alpha|+|\beta|<N+m}S_{\alpha\beta}x^\alpha\hat{y}^\beta,$$

so that $\tilde{h}=h\circ\varphi_1^{-1}$ satisfies $\tilde{h}_{\alpha\beta}=0$ for all $\alpha\neq\beta$ and $|\alpha|+|\beta|<N+m$. We know that $\tilde{h}_{\alpha\beta}=h_{\alpha\beta}+D_{\alpha\beta}(h)$, where $D_{\alpha\beta}(h)$ depends on $h_{\alpha'\beta'}$ with $|\alpha'|+|\beta'|<|\alpha|+|\beta|$ and on $1/(\lambda\cdot(\alpha''-\beta''))$ with $|\alpha''|+|\beta''|<|a|+|b|$, $\alpha''\neq\beta''$. Apply a formal symplectic map φ_2 of the form (2.1) with

$$S(x,\hat{y})=\sum_{\alpha\neq\beta,|\alpha|+|\beta|\geq N+m}S_{\alpha\beta}x^\alpha\hat{y}^\beta,$$

so that $\tilde{h}\circ\varphi_2^{-1}$ is in the Birkhoff normal form. By (2.5), in which h is actually $\tilde{h}$ now, we can write (with abuse of notation for $Q_{ab}(h)$)

$$\tilde{h}_{\alpha\alpha}+C_{N+m}(\tilde{h})=h_{\alpha\alpha}+\hat{Q}_{ab}(h),$$
$$\tilde{h}_{ab}+Q_{ab}(\tilde{h})=h_{ab}+Q_{ab}(h),$$
$$\tilde{h}_{ab}A_{N+m}(\tilde{h})+\tilde{h}_{ba}B_{N+m}(\tilde{h})=h_{ab}A_{ab}(h)+h_{ba}B_{ab}(h)+C_{ab}(h)$$

for $C_{ab}(h)=D_{ab}(\tilde{h})A_{N+m}(\tilde{h})+D_{ba}(\tilde{h})B_{N+m}(\tilde{h})$, $A_{ab}(h)=A_{N+m}(\tilde{h})$, $B_{ab}(h)=B_{N+m}(\tilde{h})$.

We have obtain (2.6), via the above normalizing map $\varphi_2\varphi_1$. On the other hand the Birkhoff normal form $\hat{h}$, with the same quadratic form as h, is independent of the normalizing map. In other words, the right-hand side of (2.6) is independent of φ. Since each $\hat{h}_{\alpha\alpha}$ is a polynomial with integer coefficients in variables $h_{\alpha'\beta'}$, $\frac{1}{\lambda\cdot(\alpha''-\beta'')}$, we conclude that each term in (2.6) depends only on h and is a polynomial in the sought form. □

We now restrict ourselves to $|h_{\alpha\beta}|\leq 2$ for all α,β. Then we have

$$\max\{|Q_{ab}|,|Q_{ba}|,|A_{ab}|,|B_{ab}|,|C_{ab}|,\hat{Q}_{ab}|\}\leq\delta_{ab}(\lambda)^{-\tau_{ab}}, \tag{2.7}$$

where $\tau_{ab}>1$ is a constant independent of λ and

$$\delta_{ab}(\lambda)=\min\{\frac{1}{2},|\lambda\cdot(\alpha-\beta)|\colon\alpha\neq\beta,|\alpha|+|\beta|\leq|a|+|b|,(\alpha,\beta)\neq(a,b),(b,a)\}.$$

Put $\lambda_2=1$. Notice that for $a=(N,0),b=(0,m)$, one has $|a-b|=|a|+|b|$. Thus, we can choose an irrational $\lambda_1\in(0,1)$ so that

$$|(a-b)\cdot\lambda|=|N\lambda_1-m|<\frac{\delta_{ab}(\lambda)^{\tau_{ab}}}{100(N+m)!},\quad a=(N,0),b=(0,m) \tag{2.8}$$

holds for a sequence $(N,m)=(N_j,m_j)$ with N_j,m_j being positive integers. We may assume that $N_{j+1}+m_{j+1}>2(N_j+m_j)$. Put $a_j=(N_j,0)$, $b_j=(0,m_j)$.

We now complete the proof of the theorem.

We construct h for the case $\kappa=0$ first. We shall find h whose coefficients $h_{\alpha\beta}$ are real and satisfy the extra condition $h_{\alpha\beta}=h_{\beta\alpha}$. Put $h_{\alpha\beta}=0$ for all α,β with $|\alpha|+|\beta|>2$ and $(\alpha,\beta)\neq(a_j,b_j),(b_j,a_j)$. Inductively, we shall choose $h_{a_jb_j}=h_{b_ja_j}=0,2$, or -2 as follows. Notice that if u_0,v_0 are real and $|u_0v_0|<1$, then either $(u_0+2)(v_0+2)\geq 2$ or

$(u_0-2)(v_0-2)\geq 2$; otherwise, we would have both $u_0+v_0<-1/2$ and $u_0+v_0>1/2$, which is a contradiction. Therefore for two real numbers u_0, v_0, choosing (u,v) among $(0,0)$, $(2,2)$ and $(-2,-2)$ yields $|(u_0+u)(v_0+v)|\geq 1$. This shows that we can find $h_{a_jb_j}=h_{b_ja_j}=0$, 2 or -2, so that

$$|(h_{ab}+Q_{ab}(h))(h_{ba}+Q_{ba}(h))|\geq 1, \quad a=a_j, b=b_j. \tag{2.9}$$

Here, we already used $N_{j+1}+m_{j+1}>2(N_j+m_j)$, which implies that if (2.9) holds for $a=a_j, b=b_j$ then it remains true no matter how a_{j+1}, b_{j+1} are chosen. Now (2.6)-(2.9) imply that for $(N,m)=(N_j,m_j)$ and $|\lambda_1 N_j-1|>1$ we have

$$|\hat{h}_{\alpha\alpha}|>\frac{m^2|\lambda_1 N-1|}{2|\lambda\cdot(a-b)|^2}>(N+m)!, \quad \alpha=(N,m-1).$$

This shows the divergence of $\hat{h}$.

We now construct h for the case $\kappa=2=n$, via restricting the complexification of h to a suitable totally real subspace of $\mathbf{C}^4$.

For the above analytic real function $h(x,y)$ on $\mathbf{R}^2\times\mathbf{R}^2$, its complexification, denoted by $h(z,w)$, is holomorphic near $0\in\mathbf{C}^2\times\mathbf{C}^2$. Let φ be a formal symplectic map of $\mathbf{R}^4$, which is tangent to the identity, so that $h\circ\varphi^{-1}(x,y)=g(x_1y_1,x_2y_2)$ is in the normal form. Since φ preserves $\omega=dx_1\wedge dy_1+dx_2\wedge dy_2$, its complexification, still denoted by φ, preserves $\omega^c=dz_1\wedge dw_1+dz_2\wedge dw_2$.

Let $L(\xi,\eta)=(\xi+i\eta,\xi-i\eta)$. Notice that $L^*\omega^c=-2i(d\xi_1\wedge d\eta_1+d\xi_2\wedge d\eta_2)$. Thus $\psi=L^{-1}\varphi L$ preserves $d\xi_1\wedge d\eta_1+d\xi_2\wedge d\eta_2$. Also $\tilde{h}\circ\psi^{-1}(\xi,\eta)=g(\xi_1^2+\eta_1^2,\xi_2^2+\eta_2^2)$ for $\tilde{h}=h\circ L$. In other words, $\tilde{h}\circ\psi^{-1}$ is a (formal holomorphic) Birkhoff normal form with respect to the holomorphic symplectic 2-form $d\xi_1\wedge d\eta_1+d\xi_2\wedge d\eta_2$. Notice that the quadratic form of $\tilde{h}$ is now $\lambda_1(\xi_1^2+\eta_1^2)+\lambda_2(\xi_2^2+\eta_2^2)$. Let e be the restriction of $\tilde{h}$ on $\mathbf{R}^2\times\mathbf{R}^2\colon \xi=\overline{\xi}, \eta=\overline{\eta}$. Since $h_{\alpha\beta}=\overline{h}_{\beta\alpha}$ by construction, then e is real-valued. Thus $e(\xi,\eta)$ is an analytic real function of the form $\lambda_1(\xi_1^2+\eta_1^2)+\lambda_2(\xi_2^2+\eta_2^2)+O(3)$, while $L^*\omega^c$, restricted to $\mathbf{R}^2\times\mathbf{R}^2\colon \xi=\overline{\xi}, \eta=\overline{\eta}$, is a constant multiple of the standard symplectic real 2-form. Therefore $\tilde{h}\circ\psi^{-1}$, restricted to $\xi=\overline{\xi}, \eta=\overline{\eta}$, is a real Birkhoff normal form of e; since h diverges, one readily sees the divergence of the restriction.

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